lecture 23

Lec 23

4.3 Evaluating Definite Integral

last class, we used the definition of definite integrals as a limit of Riemann sums, to compute integrals and it was tedious and difficult.

Newton found an easier way (and so did Leibniz independently) and it used the antiderivative.

Evaluation Theorem

If f is continuous on $[a,b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$,

where F is any antiderivative of f, that is, $F' = f$. $\square$

It says to find definite integral of a continuous function, we find it's antiderivative (any one will work) and evaluate it at the endpoints. (crazy how that complicated sum is just antiderivative an substract)

Ex $\int_0^1 x^2\, dx$

Then $f(x) = x^2$ is continuous on $[0,1]$

and it's antiderivate is $F(x) = \frac{x^3}{3}$ (one of it)

Then, $F(1) - F(0) = \frac{(1)^3}{3} - \frac{(0)^3}{3} = \frac{1}{3}$

Pick another antiderivative

$F(x) = \frac{x^3}{3} + 4$

Then $F(1) - F(0) = \frac{1^3}{3} + 4 - \left(\frac{0^3}{3} + 4\right) = \frac{1}{3} + 4 - 4 = \frac{1}{3}$

Ex $\int_0^3 (x^3 - 6x)\, dx$

$f(x) = x^3 - 6x$ is continuous on $[0,3]$

$F(x) = \frac{x^4}{4} - \frac{6x^2}{2} = \frac{x^4}{4} - 3x^2$

Then, $\int_0^3 (x^3 - 6x)\,dx = F(3) - F(0) = \left(\frac{3^4}{4} - 3.3^2\right) - \left(\frac{0^4}{4} - 3 0^2\right)$

$= \frac{81}{4} - 27 = -\frac{27}{4}$

INDEFINITE INTEGRALS

We want a convenient notation for antiderivatives that make them easy to work with.

Given the relation betn antiderivatives and definite integrals, given by the Evaluation Thm, the notation $\int f(x)dx$ is traditionally used for antiderivative of f, called an indefinite integral.

So,

$$\int f(x) = F(x) \quad \text{means} \quad F'(x) = f(x) \text{ or that } F \text{ is the antiderivative of } f.$$

Imp $\int_a^b f(x)\,dx$ is a number, $\int f(x)\,dx$ is a function (or family of functions)

Thm If f is a continuous function,

$$\int_a^b f(x)\,dx = \int f(x)\,dx \Big]_a^b \longrightarrow \text{Rewriting the Evaluation Theorem.}$$

Table of Indefinite Integrals

$\int c f(x)\,dx = c\int f(x)\,dx$, $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

$\int K\,dx = Kx + C$, K is a constant , $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

$\int \sin x\,dx = -\cos x + C$, $\int \cos x\,dx = \sin x + C$

$\int \sec^2 x\,dx = \tan x + C$, $\int \csc^2 x\,dx = -\cot x + C$

$\int \sec x \tan x\,dx = \sec x + C$, $\int \csc x \cot x\,dx = -\csc x + C$

Ex Find the general indefinite integral

$$\int (10x^4 - 6\csc^2 x)\,dx = \int 10x^4\,dx - \int 6\csc^2 x\,dx$$

$$= 10\int x^4\,dx - 6\int \csc^2 x\,dx = 10\frac{x^5}{5} - 6(-\cot x) + C$$

$$= 2x^5 + 6\cot x + C .$$

Ex Evaluate

$$\int_1^4 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\, dt$$

$$= \int_1^4 2 + t^{1/2} - t^{-2}\, dt$$. Since $2 + t^{1/2} - t^{-2}$ is continuous on $[1, 4]$

$$= 2t + \frac{t^{3/2}}{3/2} + \frac{t^{-1}}{-1} \Big]_1^4$$

$$= 2t + \frac{2t^{3/2}}{3} + \frac{1}{t} \Big]_1^4$$

$$= \left(2.4 + \frac{2.4^{3/2}}{3} + \frac{1}{4}\right) - \left(2.1 + \frac{2.(1)^{3/2}}{3} + \frac{1}{1}\right)$$

$$= \left(8 + \frac{16}{3} + \frac{1}{4}\right) - \left(3 + \frac{2}{3}\right)$$

$$= \left(5 + \frac{14}{3} + \frac{1}{4}\right) = \left(\frac{60 + 56 + 3}{12}\right) = \frac{119}{12}$$

Application

Evaluation Thm says if f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) \text{ , where } F' = f.$$

Rewrite it as

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

So what do we Know about $F'(x)$? It represents the rate of change of $y = F(x)$ with respect to x.

On the other hand, $F(b) - F(a)$ is the actual change in $y = F(x)$ as x changes from a to b. This can be expressed in the following way:

NET CHANGE THM

The integral of rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

- This principal can be used to compute various examples.

- If $V(t)$ is the volume of water in a reservoir at time t, then it's derivative $V'(t)$ is the rate at which water flows into the reservoir at time t. So,

$$\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1)$$ is the change in the amount of water in the reservoir between time t_1 and t_2.

- If The rate of growth of a population is $\frac{dn}{dt}$, then

$$\int_{t_1}^{t_2} \left(\frac{dn}{dt}\right) dt = n(t_2) - n(t_1)$$ is the net change in population during the time period from t_1 to t_2

- If an object moves along a straight line with position function $s(t)$, then it's velocity is $v(t) = s'(t)$, so

$$\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$$ is the net change of position, called displacement, of the object during the time period from t_1 to t_2.

Ex A particle moves along a line so that it's velocity at time t is

$v(t) = t^2 - t - 6$ (in m/s).

Find the displacement of the particle during the time period

$1 \leqslant t \leqslant 4$

Sol^n The displacement is change in position

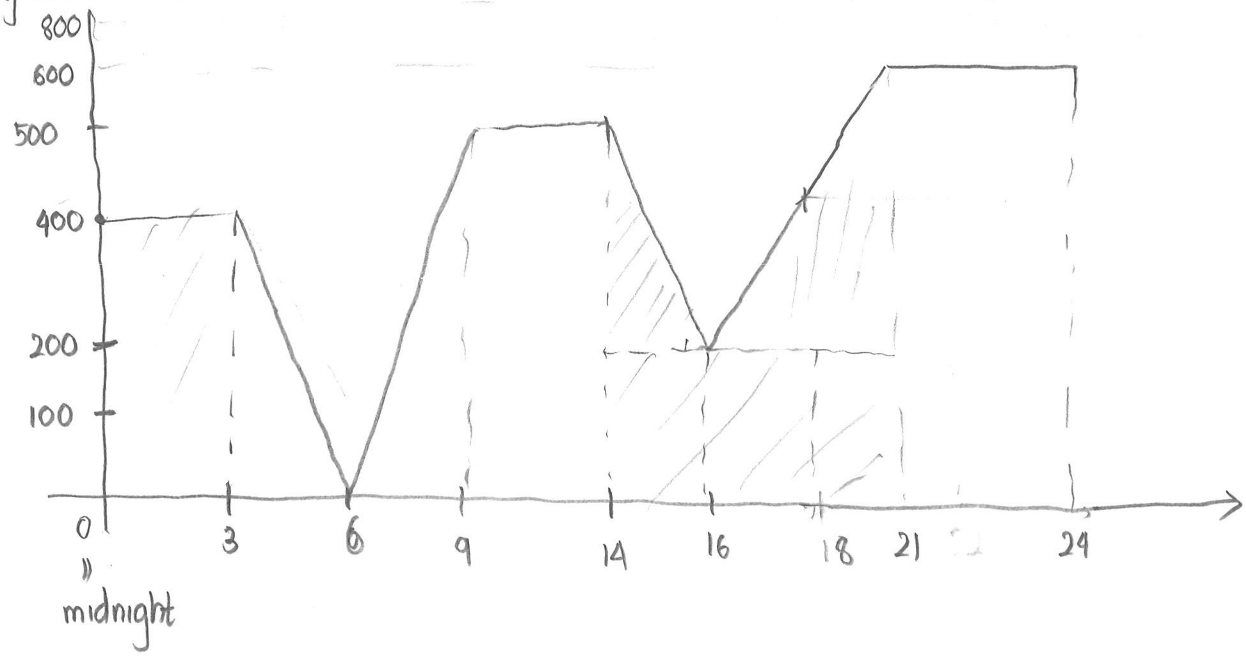
$$s(4) - s(1) = \int_1^4 v(t)\,dt = \int_1^4 (t^2 - t - 6)\,dt$$

$$= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_1^4 = \left(\frac{4^3}{3} - \frac{4^2}{2} - 6.4 \right) - \left(\frac{1^3}{3} - \frac{1^2}{2} - 6.1 \right)$$

$$= \left(\frac{64}{3} - 8 - 24 \right) - \left(\frac{1}{3} - \frac{1}{2} - 6 \right)$$

$$= \left(\frac{-32}{3} + \frac{37}{6} \right) = \left(\frac{-64}{6} + \frac{37}{6} \right) = \frac{-27}{6} = \frac{-9}{2} = -4.5$$ ☒

<u>Ex</u> Let $P(t)$ denote the power consumption in the city of Miami for a day in November (P is measured in Mega Watts and t in hours starting at midnight) . Let the graph of $P(t)$ be given below .



Use the fact the rate of change of energy is power to estimate the energy consumed on that day ?

<u>Sol^n</u> Power is the rate of change of energy ; that means $P(t) = E'(t)$.
To find the total energy consumed in the day we use the Net Change Theorem

$$\int_0^{24} P(t)\,dt = \int_0^{24} E'(t)\,dt = \underbrace{E(24) - E(0)}_{\substack{\text{total amount} \\ \text{of energy used in the day}}} .$$

Given the graph, of $P(t)$, to find

$$\int_0^{24} P(t)\,dt$$, we just need to find the area under the curve

$1200 + 600 + 750 + 2500 + 1400 + 300 + 1000 + 1800 = 9550$ MV-hr.